Does Lyndon's Lenth Function Imply

Imply the Universal Theory of Free Groups?

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ABSTRACT. This note shows that every model of the universal theory of the non-Abelian free groups admits a Lyndon length function. The question is then posed as to whether the model class of the universal theory of the non-Abelian free groups is precisely the class of such groups. The authors have subsequently given a negative answer to this question.

Definitions and notation will be that of Bell and Slomson [3], and of Gaglione and Spellman [6], [7] and [8].

Let L be a first-order language with equality. Two L-structures A and B *have the same universal theory* just in case they satisfy precisely the same universal sentences (and therefore also precisely the same existential sentences) of L. If B is an L-structure, let $\mathrm{Th}(B) \cap (\forall \cup \exists)$ be the set of all universal and existential sentences of L true in B. Evidently the L-structure A has the same universal theory as B if and only if A is a model of $\mathrm{Th}(B) \cap (\forall \cup \exists)$.

If A is a substructure of the L-structure B, then a necessary and sufficient condition that A and B have the same universal theory is that there be a model *A of
Th(A) $\cap$ $(\forall \cup \exists)$ such that $A \subseteq B \subseteq *$A. This in turn is equivalent to the existence of an index set I and an ultrafilter D on I such that B is embeddable in the ultrapower A^I/D. A different necessary and sufficient condition that an L-structure B and a substructure A have the same universal theory is that A and B satisfy precisely the same primitive sentences of L. (See $[3], Ch.9$.)

Let A be a non-trivial, torsion-free, Abelian group. Let $<$ be a strict linear order on A such that for arbitrary $(a, b, c) \in A^3$ we have $a + c < b + c$ whenever $a <$ b. Then the ordered pair $\Lambda = (A, <)$ is an *ordered Abelian group.* A non-trivial, torsion-free, Abelian group A is *orderable* provided there is at least one strict linear order $<$ such that $\Lambda = (A, <)$ is an ordered Abelian group. It is well-known that every non-trivial, torsion-free, Abelian group is

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orderable. None the less, we'll present an argument to that effect in this paper. We shall also show the well-known result that this class of Abelian groups is the class having the same universal theory as $\mathbb{Z}$; moreover, we show that every model of the universal theory of the non-Abelian free groups admits a Lyndon length function.

We now specify two first-order languages with equality. L_o shall contain a binary operation symbol $\cdot$, a unary operation symbol $^{-1}$and a constant symbol $1.L$ shall contain a binary operation symbol $+$, a unary operation symbol -, a constant symbol 0 and a binary relation symbol $<$ $.L_o$ shall be the *language of group theory* and L shall be the *language of ordered Abelian groups.* A primitive sentence of L_o is one of the form

$$\exists(\mid (p_i() = P_i()) \mid \mid (q_j() \neq Q_j()))$$

where $\overline{x}$ is a tuple of variables and the $p_i(x), P_i(x), q_j(x)$ and $Q_j(x)$ are terms of L_o.

In case the L_o-structures we are considering are groups
this type of sentence may be simplified to one of the form

$$\exists(\mid (p_i() = 1) \mid \mid (q_j() \neq 1))$$

where $\overline{x} = (x_1, \dots, x_m)$ is a tuple of distinct variables and the $p_i(\overline{x})$ and $q_j(\overline{x})$ are words on $\{x_1, \dots, x_m\} \cup \{x_1, \dots, x_m\}$.

LEMMA 1. *A non-trivial Abelian group A has the same universal theory as* $\mathbb{Z}$ *if and only if A is torsion-free.*

PROOF. One implication is trivial. Assume A is a non-trivial, torsion-free, Abelian group to show that A is a model of
$\mathrm{Th}(\mathbb{Z}) \cap (\forall \cup \exists)$. We write our groups additively here. Let a be a non-zero element of A and put $A_0 =< a >\cong \mathbb{Z}$. It will suffice to show that A_0 satisfies every primitive sentence true in A. To that end consider the system

$$(*)$$

of equations and inequations. Suppose (*) has a solution $(x_1, \dots, x_m) = (a_1, \dots, a_m)$ in A. It suffices to show that (*) has a solution in A_0. Let B be the subgroup of A generated by $\{a_1, \ldots, a_m\}$. If $B = 0$, then $(x_1, \ldots, x_m) = (0, \ldots, 0)$ is a solution to (*) in A_0. We may therefore assume $B \neq 0$. So B is then a non- trivial, finitely generated, torsion free, Abelian group. Thus, B is free Abelian of some finite rank $r \geq 1$. But it was shown in Gaglione and Spellman [7] that $\mathbb{Z}^r$ and $\mathbb{Z}$ have the same universal theory. Therefore (*) has a solution in A_0.■

DEFINITION 1 (Lyndon [12]). Let G be a (multiplicatively written) group. Let $\Lambda = (A, <)$ be an (additively written) ordered Abelian group. Let $\lambda : G \longrightarrow A, g \longmapsto \mid g \mid$ be a function and let
$2c : G^2 \longrightarrow A$ be defined by $(g_1, g_2) \longmapsto \mid g_1 \mid + \mid g_2 \mid - \mid g_1 g_2 \mid$. The the ordered triple (G, Λ, λ) is a *normed group* provided the following six axioms are satisfied:

$(A0) x \neq 1$ implies $\mid x \mid < \mid x^2 \mid$

$(A1) \mid x \mid \geq 0$ and $\mid x \mid = 0$ iff x = 1

$(A2) \mid x^{-1} \mid = \mid x \mid$

$(A3) 2c(x, y) \geq 0$

$(A4) 2c(x, y) > 2c(x, z)$ implies $2c(y, z) = 2c(x, z)$

(CO)$2c(x, y) \equiv 0 (\mathrm{mod} 2A)$

A group G is *normable* provided there is at least one ordered Abelian group $\Lambda = (A, <)$ and at least one map $\lambda : G \longrightarrow A$ such that (G, Λ, λ) is a normed group.

REMARK. The axioms are not independent. Chiswell [5] has shown that (A3) is a consequence of $(A2), (A4)$ and the following $(A1') \mid 1 \mid = 0$.

If S is a set of sentences of a first-order language L with equality, let $\mathbb{M}(S)$ be the model class of S.

THEOREM 1. *Let L be a first-order language with equality. Let X be a class of L-structures. Then the following three properties form a set of necessary and sufficient conditions that X be of the form $\mathbb{M}(S)$ for at least one set S of sentences of L*:

(i) *X is closed under isomorphism.*

(ii) *X is closed under the formation of ultraproducts.*

(iii) *If* cX *is the class of all* L*-structures not in* X*, then* cX *is closed under the formation of ultrapowers.*

Thoerem 1 is a deduction of Theorem 3.10, Chapter 7 of [3] without assuming (G.C.H.) using the Keisler-Shelah Theorem. (See [3] and [15].)

Although "the" norm is not generally defined in L_o, norms are internal in the sense that they extend to ultraproducts. It is then straightforward to deduce -

COROLLARY. *The class of all non-Abelian, normable groups is the model class* $\mathbb{M}(\Theta)$ *of some set* Θ *of sentences of* L_o.

It is known that the non-Abelian free groups have the same universal theory (see [7]). Thus, if F_2 is free of rank 2 and $\Phi = \mathrm{Th}(F_2 \cap (\forall \cup \exists)$, then every non-Abelian, free group is a model of Φ. It is not difficult to convince oneself that the models of Φ are precisely those non-Abelian groups embeddable in some ultrapower of F_2, since one can easily show that every model of Φ contains a copy of F_2. But if $F_2 =< a_1, a_2 >$, then the length function with respect to the free basis $\{a_1, a_2\}$,
$\lambda : \mathrm{F}_2 \longrightarrow \mathbb{Z}$ induces a length function $*\lambda : F_2/D \longrightarrow \mathbb{Z}^I/D$ making $(F_2/D, (\mathbb{Z}^I/D, <), *\lambda)$ into a normed group whenever I is an index set and D is an ultrafilter on I. The restriction of $*\lambda$ to the subgroup G makes G into a normed group. Thus, every model of Φ is also a model of Θ. In symbols -

THEOREM 2. $\mathbb{M}(\Phi) \subseteq \mathbb{M}(\Theta)$.

Brignole and Ribeiro [4] have given a proof of a theorem of Gurevic and Kokorin asserting that any two ordered Abelian groups have the same universal theory. Since any ordered Abelian group $\Lambda = (A, <)$ contains an ordered subgroup isomorphic to $(\mathbb{Z}, <)$ it follows that every ordered Abelian group Λ is embeddable in some ultrapower

$$(\mathbb{Z}, <)^I/D = (\mathbb{Z}^I/D, <).$$

Thus, every normed group admits a norm with values in an ordered Abelian group of the form $(\mathbb{Z}^I/D, <)$.

Let L be a first-order language with equality. A sentence of L of the form $\forall\overline{x}\exists\overline{y}\phi(\overline{x},\overline{y})$ where $\overline{x}$ and $\overline{y}$ are disjoint tuples of variables, $\phi(\overline{x},\overline{y})$ contains no quantifiers and $\phi(\overline{x},\overline{y})$ contains free at most the variables in $\overline{x}$ and $\overline{y}$ is a *universal-existential* sentence of L. Any sentence of L logically equivalent to a universal-existential sentence of L is *a*$\blacksquare_2$*-sentence* of L. Since vacuous quantifications are permitted, every universal sentence of L and every existential sentence of L is also *a*$\blacksquare_2$-sentence of L.

THEOREM 3. Θ *may be taken to be a set of* $\blacksquare_2$*-sentences of* L_o.

PROOF. In view of Theorem 2, p. 279 of Grätzer [10], it suffices to show that the union $G = \bigcup_{n<\omega} G_n$ of a chain $(G_n)_{n<\omega}$ of non- Abelian, normable subgroups $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$ is normable. To that end suppose that $(G_n, \Lambda_n, \lambda_n)$ is a normed group. Let D be a non-principal ultrafilter on ω and let $\Lambda =$
$(\prod_{n<\omega} \Lambda_n)/D$ be the ultraproduct of the family $(\Lambda_n)_{n<\omega}$ of ordered Abelian groups with respect to the ultrafilter D. Let $\Lambda = (A, <)$. For each $g \in G$, let $\deg(g) = \min\{n \in \omega \mid g\epsilon G_n\}$. Finally, let
$\lambda : G \longrightarrow A$ be given by g $\mapsto L_g/D$ where

Then it is straightforward to verify that (G, Λ, λ) is a normed group.■

QUESTION. *Is* $\mathbb{M}(\Phi) = \mathbb{M}(\Theta)$?

Equivalently: *Does every non-Abelian, normable group have the same universal theory as the non-Abelian free groups?*

In view of (5.3), (5.4)(6.4) of Alperin and Bass [1], we may also pose the

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QUESTION. *Let* G *be a non-Abelian group. Is it the case that* G *is a model of* Φ *if and only if there is an ordered Abelian group* Λ *and a*Λ*-tree* T *such that* G *acts freely on* Λ *without inversions?*
(i.e., if and only if G *is tree-free in the sense of Bass - see [2].)*

Addendum

Since the original preparation of the manuscript, the authors have learned of [14]. In that work, Remeslennikov also shows that every model of Φ is normable. Moeover, a negative answer to our question is given independently in [9] and in [14].

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